Assignment 1 - Math Camp

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Problem 1. If x is an even integer, then x^2 is even.

Suppose x is an even integer.
Thus $x = 2k$ for some $k \in \mathbb{Z}$
Observe:
$x^2 = x \cdot x$
$= (2k)(2k)$
$= 2(2k^2)$ where $2k^2 \in \mathbb{Z}$ ⋆
Hence x^2 is an even integer ■

Problem 2. Suppose $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.

$x^2 + 5y = y^2 + 5x$
$x^2 - y^2 = 5(x - y)$
$(x + y)(x - y) = 5(x - y)$
Consider the following cases:

Case 1: Suppose $x - y = 0$
Then it follows that $x = y$

Case 2: Suppose $x - y \neq 0$
Dividing $(x + y)(x - y) = 5(x - y)$ by $(x - y)$, we get:
$(x + y) = 5$

Thus $x = y$ or $x + y = 5$ ■

Problem 3. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

Using proof by contraposition, assume a or b is even.
Consider the following cases:

Case 1: Suppose a is even and b is even
Thus $a = 2k$ and $b = 2l$ for some $k, l \in \mathbb{Z}$
Thus we see that for $a^2(b^2 - 2b)$:
$= (2k)^2((2l)^2 - 2(2l))$
$= 4k^2(4l^2 - 4l)$
$= 2(8k^2l^2 - 8k^2l)$ where $8k^2l^2 - 8k^2l \in \mathbb{Z}$
Thus $a^2(b^2 - 2b)$ is even

Case 2: Suppose a is even and b is odd

⋆Rule 4.1 from Book of Proof by Richard Hammack: multiplication, addition, and subtraction of integers result in integers.

Thus $a = 2k + 1$ and $b = 2l + 1$ for some $k, l \in \mathbb{Z}$
Thus we see that for $a^2(b^2 - 2b)$:
$= (2k)^2((2l + 1)^2 - 2(2l + 1))$
$= 4k^2(4l^2 - 1)$
$= 2(8k^2l^2 - 2k^2)$ where $8k^2l^2 - 2k^2 \in \mathbb{Z}$
Thus $a^2(b^2 - 2b)$ is even

Case 3: Suppose a is odd and b is even
Thus $a = 2k + 1$ and $b = 2l$ for some $k, l \in \mathbb{Z}$
Thus we see that for $a^2(b^2 - 2b)$:
$= (2k + 1)^2((2l)^2 - 2(2l))$
$= (4k^2 + 4k + 1)(4l^2 - 4l)$
$= 2(8k^2l^2 - 8k^2l + 8kl^2 - 8kl + 2l^2 - 2l)$ where $8k^2l^2 - 8k^2l + 8kl^2 - 8kl + 2l^2 - 2l \in \mathbb{Z}$
Thus $a^2(b^2 - 2b)$ is even

Case 4: Suppose a is odd and b is odd
Thus $a = 2k$ and $b = 2l + 1$ for some $k, l \in \mathbb{Z}$
Thus we see that for $a^2(b^2 - 2b)$:
$= (2k + 1)^2((2l + 1)^2 - 2(2l + 1))$
$= (4k^2 + 4k + 1)(4l^2 - 4l)$
$= 2(8k^2l^2 - 8k^2l + 8kl^2 - 8kl + 2l^2 - 2l)$ where $8k^2l^2 - 8k^2l + 8kl^2 - 8kl + 2l^2 - 2l \in \mathbb{Z}$
Thus $a^2(b^2 - 2b)$ is even

Therefore in any case (whether a be even or odd, and whether b be even or odd), $a^2(b^2 - 2b)$ is even
■

Problem 4. Prove that $\sqrt{3}$ is irrational

Suppose by contradiction that $\sqrt{3}$ is rational
Thus $\sqrt{3} = \frac{a}{b}$ where $\frac{a}{b}$ is fully reduced and $a, b \in \mathbb{Z}$
$3b^2 = a^2$
Thus a^2 is a multiple of 3
Since a^2 is a multiple of 3, then a is a multiple of 3
so $3b^2 = (3k)^2$ for some $k \in \mathbb{Z}$
Thus $b^2 = 3k^2$ so b is a multiple of 3
But this is the contradiction as we assumed that $\frac{a}{b}$ was fully reduced
Thus $\sqrt{3}$ is irrational ■

Problem 5. Suppose $x \in \mathbb{Z}$. Then x is even if and only if 3x+5 is odd.

$\Rightarrow$) Suppose x is even
$x = 2k$ for some $k \in \mathbb{Z}$
So $3(2k) + 5$
$= 6k + 5$
$= 2(3k) + 4 + 1$
$= 2(3k + 2) + 1$ where $3k + 2 \in \mathbb{Z}$
Thus $3x + 5$ is odd

$\Leftarrow$) Suppose by contraposition that x is odd
So $x = 2b + 1$ for some $b \in \mathbb{Z}$
Thus $3x + 5 = 3(2b + 1) + 5$
$= 6b + 8$
$= 2(3b + 4)$ where $3b + 4 \in \mathbb{Z}$

Thus $3x + 5$ is even
Therefore if $3x + 5$ is odd, then x is even ■

Problem 6. If p and q are positive integers, then $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\} \neq \emptyset$

So $p, q \in \mathbb{N}$
Thus $pq \in \{pn : n \in \mathbb{N}\}$
Also $pq \in \{qn : n \in \mathbb{N}\}$
Therefore $pq \in \{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\}$
Hence $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\} \neq \emptyset$ ■

Problem 7. If A and B are sets in the universal set U, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Show $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$
Let $a \in \overline{A \cup B}$
Thus $a \notin A \cup B$
$\neg(a \in A \cup B)$
$\neg(a \in A \vee a \in B)$
$\neg(a \in A) \wedge \neg(a \in B)$
$a \notin A \wedge a \notin B$
$a \in \overline{A} \cap \overline{B}$
So $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Show $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$
Let $b \in \overline{A} \cap \overline{B}$
$b \in \overline{A} \wedge b \in \overline{B}$
$b \notin A \wedge b \notin B$
$\neg(b \in A) \wedge \neg(b \in B)$
$\neg(b \in A \vee b \in B)$
$\neg(b \in A \cup B)$
$b \notin A \cup B$
Thus $b \in \overline{A \cup B}$
So $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$

Thus $\overline{A \cup B} = \overline{A} \cap \overline{B}$ ■

Problem 8. For each $a \in \mathbb{R}$, let $\{A_a = \{(x, a(x^2 - 1))\} \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Prove that $\bigcap_{a \in \mathbb{R}} A_a = \{(-1, 0), (1, 0)\}$

First let's prove that $\bigcap_{a \in \mathbb{R}} A_a \subseteq \{(-1, 0), (1, 0)\}$ Let $(y, z) \in \bigcap_{a \in \mathbb{R}} A_a$
In other words $(y, z) \in A_a \forall a \in \mathbb{R}$
Consider the following cases:

Case 1: $a = 0$
We see that $A_0 = \{(x, 0(x^2 - 1))\}$
Thus $A_0 = \{x, 0\}$

Case 2: $a \in \mathbb{R} - \{0\}$
$A_a = \{(x, a(x^2 - 1))\}$
In order to show that $\bigcap_{a \in \mathbb{R}} A_a \subseteq \{(-1, 0), (1, 0)\}$
Let's show that $z = 0$, or in other words $a(x^2 - 1) = 0$
Since $a \neq 0$, $x^2 - 1 = 0$

$x^2 = 1 \Rightarrow x = \pm 1$
Thus $(y, z) = (1, 0)$ or $(-1, 0)$
So $\bigcap_{a \in \mathbb{R}} A_a \subseteq \{(-1, 0), (1, 0)\}$

Now prove $\bigcap_{a \in \mathbb{R}} A_a \supseteq \{(-1, 0), (1, 0)\}$
Set $x = 1$, we see $A_a = (1, a(1^2 - 1))$
$= (1, 0)$
Now set $x = -1$, we see $A_a = (-1, a((-1)^2 - 1))$
$= (-1, 0)$
Thus, $\bigcap_{a \in \mathbb{R}} A_a \supseteq \{(-1, 0), (1, 0)\}$

Therefore, $\bigcap_{a \in \mathbb{R}} A_a = \{(-1, 0), (1, 0)\}$ ■

Problem 9. If $x, y \in \mathbb{R}$, then $|x + y| = |x| + |y|$

We present a counterexample:
Let $x = -y \neq 0$
We see that $|x + y| = 0$, and that $|x| + |y| = 2|x| = 2|y| \neq 0$
Thus $|x + y| \neq |x| + |y|$ ■

Problem 10. Let f be a function from A to B and let $E, F \subseteq A$

a) Prove that the rational preference relation, $\succeq$, is reflexive.

Reflexivity of $\succeq$ follows directly from completeness
Completeness means that $\forall x, y \in A$, we know that $x \succeq y$ or $y \succeq x$
Let $x = y$, then we see that $x \succeq x$ ■

b) Defining the indifference relation, $\sim$, as $x \sim y$ if $[x \succeq y \wedge y \succeq x]$, prove that $\sim$ is an equivalence relation.

We know $x \sim y$ if $x \succeq y \wedge y \succeq x$
We are required to prove that $\succeq$ is an equivalence relation, or that $\sim$ is reflexive, symmetric, and transitive

Reflexivity:
We know that $\succeq$ is reflexive, thus $x \succeq x$.
If we apply the definition of $\sim$, we see that $x \succeq x$ (so $x \succeq x$ and $x \succeq x$) implies $x \sim x$
Thus the indifference relation, $\sim$, is reflexive

Symmetric:
We are required to show that $x \sim y \iff y \sim x$
By definition of the indifference relation, $x \sim y \iff x \succeq y \wedge y \succeq x$
So we know that $x \succeq y \wedge y \succeq x$
We can rearrange this statement: $y \succeq x \wedge x \succeq y$
Now $y \succeq x \wedge x \succeq y \Rightarrow y \sim x$
Similarly, we can show that $y \sim x \Rightarrow x \sim y$
Thus $\sim$ is symmetric

Transitivity
We are now required to prove if $x \sim y \wedge y \sim z \Rightarrow x \sim z$
$x \sim y \Rightarrow x \succeq y \wedge y \succeq x$
$y \sim z \Rightarrow y \succeq z \wedge z \succeq y$
By transitivity of preference relation, $\succeq$, we see that $x \succeq y \wedge x \succeq z \Rightarrow x \succeq z$

Likewise, we see that $z \succeq y \wedge y \succeq x \Rightarrow z \succeq x$
Now $x \succeq z \wedge z \succeq x \Rightarrow x \sim z$
Thus $x \sim y \wedge y \sim z \Rightarrow x \sim z$
So the indifference relation, $\sim$, is transitive

Since the indifference relation, $\sim$, is reflexive, transitive, and reflexive, $\sim$ is an equivalence relation ■

c) Define the quotient set, call it I, associated with (C, $\sim$) using set notation and prove that I is a partition of C.
$I = C/\sim = \{[x] : x \in C\}$ where $[x] = \{y \in C : x \sim y\}$
To show $I = C/\sim$ is a partition, we need to show 3 things:

1. The union of all subsets of C, in this case equivalence classes, makes up the whole set C

   We need to prove that $\bigcup I = C$
   First, let's prove $\bigcup I \supseteq C$
   By definition of equivalence class: we know $\forall x \in C : x \in [x]_\sim$
   Using DeMorgan's Laws: $\neg(\exists x \in C : x \notin [x]_\sim)$
   By the definition of unions: $\neg(\exists x \in C : x \notin \bigcup [x]_\sim)$
   Again, using Demorgan's Laws: $\forall x \in C : x \in \bigcup I$
   Thus $\bigcup I \supseteq C$

   Now let's prove $\bigcup I \subseteq C$
   This from from definition of equivalence class: $\forall X \in I : X \subseteq C$
   Thus, $\bigcup I \subseteq C$

   Therefore $\bigcup I = C$

2. Equivalence classes are disjoint

   Required to prove: $(x, y) \notin \sim \iff [x]_\sim \bigcap [y]_\sim = \emptyset$

   We first prove that $(x, y) \notin \sim \Rightarrow [x]_\sim \bigcap [y]_\sim = \emptyset$
   Suppose by contraposition that $[x]_\sim \bigcap [y]_\sim \neq \emptyset$
   Thus $\exists z$ such that $z \in [x]_\sim \bigcap [y]_\sim$
   So $z \in [x]_\sim \wedge z \in [y]_\sim$
   Therefore $x \sim z \wedge y \sim z$ by definition of equivalence class
   Since $\sim$ is symmetric, $\Rightarrow x \sim z \wedge z \sim y$
   By transitivity, $x \sim y$
   So $(x, y) \in \sim$
   Thus $(x, y) \notin \sim \Rightarrow [x]_\sim \bigcap [y]_\sim = \emptyset$

   Now we need to show that $[x]_\sim \bigcap [y]_\sim = \emptyset \Rightarrow (x, y) \notin \sim$
   Suppose by contraposition that $(x, y) \in \sim$
   We know that $x \in [x]_\sim$ and $y \in [y]_\sim$ by definition of equivalence class
   Thus $y \in [x]_\sim$ since $(x, y) \in \sim$
   So $y \in [y]_\sim \wedge y \in [x]_\sim$
   Therefore $y \in [y]_\sim \bigcap [x]_\sim$
   Hence $[x]_\sim \bigcap [y]_\sim \neq \emptyset$
   So $x \sim y \Rightarrow [x]_\sim \bigcap [y]_\sim \neq \emptyset$
   Thus $[x]_\sim \bigcap [y]_\sim = \emptyset \Rightarrow (x, y) \notin \sim$

   Combining these two findings, we get $(x, y) \notin \sim \iff [x]_\sim \bigcap [y]_\sim = \emptyset$

3. Equivalence classes are non-empty

   We need to prove that $\forall [x]_\sim \subseteq C : \exists x \in C$
   This follows from the definition of equivalence classes. We know $x \in [x]_\sim$
   Thus, it follows that $[x]_\sim \neq \emptyset$

Thus I is a partition ■

d) Interpret the equivalence classes of I and show that $(I, \succeq)$ is a partially ordered set.

The equivalence classes are like indifference sets. Each equivalence class contains elements that are indifferent to each other (and are like bundles in the same indifference set or on the same indifference curve).

To show that $(I, \succeq)$ is a partially ordered set, we need to show it satisfies reflexivity, antisymmetry, and transitivity.

Reflexivity:

Let $[x]_\sim \in I$.
We are required to prove that $[x]_\sim \succeq [x]_\sim$
As we showed earlier, $x, y \in [x]_\sim \Rightarrow x \sim y$ and $y \sim x$, $\forall x, y \in [x]_\sim$
Thus $x \succeq y \wedge y \succeq x$ from definition of $\sim$
Using transitivity (of equivalence class), we find that $x \sim y \succeq x \sim y$ (we can invoke symmetry of equivalence classes to get any combo we want, i.e $x \sim y$ or $y \sim x$)
Thus $[x]_\sim \succeq [x]_\sim$

Antisymmetry:

We are required to show that $[x]_\sim \succeq [y]_\sim \wedge [y]_\sim \succeq [x]_\sim \Rightarrow [x]_\sim = [y]_\sim$
Let $a, b \in [x]_\sim$ and $c, d \in [y]_\sim$
Thus we know that $a \sim b$, and $c \sim d$
$[x]_\sim \succeq [y]_\sim \Rightarrow a \sim b \succeq c \sim d, \ \forall a, b \in [x]_\sim$ and $\forall c, d \in [y]_\sim$
$[y]_\sim \succeq [x]_\sim \Rightarrow c \sim d \succeq a \sim b, \ \forall a, b \in [x]_\sim$ and $\forall c, d \in [y]_\sim$
$a \sim b \succeq c \sim d \wedge c \sim d \succeq a \sim b \Rightarrow a \sim b \sim c \sim d$
Thus, $a \sim b \sim c \sim d \in [x]_\sim$ and $a \sim b \sim c \sim d \in [y]_\sim$
Thus $[x]_\sim = [y]_\sim$

Transitivity:

We are required to show that $[x]_\sim \succeq [y]_\sim \wedge [y]_\sim \succeq [z]_\sim \Rightarrow [x]_\sim \succeq [z]_\sim$
Let $e, f \in [x]_\sim$ and $g, h \in [y]_\sim$, and $i, j \in [z]_\sim$
Thus we know that $e \sim f$, and $g \sim h$, and $i \sim f$
Suppose $[x]_\sim \succeq [y]_\sim \wedge [y]_\sim \succeq [z]_\sim$
Thus $e \sim f \succeq g \sim h$ and $g \sim h \succeq i \sim j$, $\forall e, f \in [x]_\sim$ and $\forall g, h \in [y]_\sim$ and $\forall i, j \in [z]_\sim$
Hence $e \sim f \succeq i \sim j$, $\forall e, f \in [x]_\sim$ and $\forall i, j \in [z]_\sim$
Therefore $[x]_\sim \succeq [z]_\sim$

Thus, $(I, \succeq)$ is a partially ordered set ■

Problem 11. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x, y) = (xy, x^3)$

Is f injective?
We will show that f is not injective
We see $f(0,b) = (0,0)\ \forall b \in \mathbb{R}$
So f is not injective

Is f surjective?
We will next show that f is not surjective
Consider $f(x,y) = (1,0)$
In order for $f(x,y) = (1,0)$, we need $xy = 1$ and $x^3 = 0$
The only way we can have $x^3 = 0$ is if $x = 0$
But if $x = 0$, $xy \neq 1\ \forall y \in \mathbb{R}$
So f is not surjective

Since f is not injective or surjective, it cannot be bijective ■

Problem 12. Show:

a) If $E \subseteq F \Rightarrow f(E) \subseteq f(F)$

Let $a \in E$
So $a \in F$ since $E \subseteq F$
We see that since $a \in E \Rightarrow f(a) = y$ for some $y \in f(E)$
But also since $a \in F \Rightarrow f(a) = y$ for some $y \in f(F)$
Thus, $f(E) \subseteq f(F)$ ■

b) $f(E \cap F) \subseteq f(E) \cap f(F)$

Let $f(E \cap F) = \{f(x) : x \in E \cap F\}$
$f(E) = \{f(x) : x \in E\}$
$f(F) = \{f(x) : x \in F\}$

Let $y \in f(E \cap F)$
$\Rightarrow y = f(e)$ for some $e \in E \cap F$
Now $e \in E \Rightarrow y = f(e) \in f(E)$
And $e \in F \Rightarrow y = f(e) \in f(F)$
Thus $y \in f(E) \cap f(F)$ ■

c) $f(E \cup F) = f(E) \cup f(F)$

Let's prove that $f(E \cup F) \subseteq f(E) \cup f(F)$
Let $y \in f(E \cup F)$
$\Rightarrow y \in \{f(x) : x \in E \cup F\}$
$\Rightarrow y = f(e)$ for some $e \in E \cup F$
$\Rightarrow e \in E$ or $e \in F$
If $e \in E$, then $y = f(e) \in f(E)$
If $e \in F$, then $y = f(e) \in f(F)$
Thus $y = f(e) \in f(E) \cup f(F)$

Let's now prove that $f(E \cup F) \supseteq f(E) \cup f(F)$
So $y \in f(E)$, thus $y = f(e)$ for some $e \in E$
or $y \in f(F)$, thus $y = f(e)$ for some $e \in F$
Thus $y = f(e)$ for some $e \in E \cup F$

Thus $y \in f(E \cup F)$
So, $f(E \cup F) \supseteq f(E) \cup f(F)$

Therefore $f(E \cup F) = f(E) \cup f(F)$ ■

d) $f(E - F) \subseteq f(E)$

Let $y \in f(E - F)$
Thus $y = f(a)$ for some $a \in (E - F)$
Hence $a \in E \wedge a \notin F$
Therefore $y = f(a)$ where $f(a) \in f(E)$
Therefore $f(E - F) \subseteq f(E)$ ■

e) $f(E \Delta F) \subseteq f(E) \Delta f(F)$

Let $b \in f(E\Delta F) = \{f(x) : x \in (E\Delta F)\}$
So $b = f(a)$ for some $a \in (E\Delta F)$
Thus $a \in (E - F) \cup (F - E)$
$a \in (E - F) \vee a \in (F - E)$
Therefore $a \in E \wedge a \notin F \vee a \in F \wedge a \notin E$

Case 1: Let $a \in E \wedge a \notin F$
If $a \in E \wedge a \notin F$ then $b = f(a) \in F(E) \wedge b = f(a) \notin f(F)$
So $b \in (f(E) - f(F))$

Case 2: Let $a \in F \wedge a \notin E$
If $a \in F \wedge a \notin E$ then $b = f(a) \in F(F) \wedge b = f(a) \notin f(E)$
So $b \in (f(F) - f(E))$

Thus $b \in (f(E) - f(F)) \cup (f(F) - f(E))$
So $b \in f(E)\Delta f(F)$
Thus $f(E\Delta F) \subseteq f(E)\Delta f(F)$ ■

Problem 13. Let $f : A \to B$. Define $a \sim_f a'$ if $\exists b \in B$ such that $a, a' \in f^{-1}(b)$

a) Show $\sim_f$ is an equivalence relation.

We need to show that $\sim_f$ is: reflexive, symmetric, and transitive.

Reflexivity:
We are required to show that $c \sim_f$ c
Suppose $c \in A$
Let $b = f(c)$ where $b \in B$
Thus we see that $c \in f^{-1}(b)$
Then by definition of $\sim_f$ any element in $f^{-1}(b)$ can be used to state $a \sim_f a'$ for any $a, a' \in f^{-1}(b)$
Applying this definition, $c \in f^{-1}(b)$, then $c \sim_f c$
Therefore $\sim_f$ is reflexive.

Symmetric:
We need to show $c \sim_f c' \Rightarrow c' \sim_f c$
Suppose $c \sim_f c'$
Thus $c, c' \in f^{-1}(b)$
By our definition of level sets, we see that $c, c' \in f^{-1}(b) \Rightarrow c' \sim_f c$

Similarly we can show that $c' \sim_f c \Rightarrow c \sim_f c'$
Thus $\sim_f$ is symmetric

Transitivity:
We need to show that $c \sim_f c', c' \sim_f c'' \Rightarrow c \sim_f c''$
Suppose $c \sim_f c'$ and $c' \sim_f c''$
Thus $c, c' \in f^{-1}(b)$ and $c', c'' \in f^{-1}(b)$
So we see that $c, c', c'' \in f^{-1}(b)$
Therefore $c \sim_f c''$
So $\sim_f$ is transitive.

Thus $\sim_f$ is an equivalence relation ■

b) Show c $\sim_f c' \Rightarrow f(c) = f(c')$

Suppose $c \sim_f c'$, thus $c, c' \in f^{-1}(b)$
Since $c, c' \in f^{-1}(b)$, then $f(c) = f(c')$

Now show $f(c) = f(c') \Rightarrow c \sim_f c'$

Suppose $b = f(c)$, thus we also know $b = f(c')$
So $f(c) = b = f(c') \Rightarrow c \sim_f c'$ ■

c) Prove that inverse images $f^{-1}(b)$ and $f^{-1}(b')$ are disjoint when $b \neq b'$

Suppose by contraposition that $f^{-1}(b)$ and $f^{-1}(b')$ are not disjoint
Since $f^{-1}(b)$ and $f^{-1}(b')$ are not disjoint we know that $f^{-1}(b) \cap f^{-1}(b') \neq \emptyset$
Thus $\exists a \in f^{-1}(b) \cap f^{-1}(b')$
So $f(a) = b$ and $f(a) = b'$
But we see that since f is a function, $f(a)$ is a singleton
Thus $b = b'$
So we see that $f^{-1}(b)$ and $f^{-1}(b')$ being not disjoint $\Rightarrow b = b'$
Therefore, $b \neq b' \Rightarrow f^{-1}(b)$ and $f^{-1}(b')$ are disjoint ■

Problem 14. Show:

a) If $G \subseteq H$, then $f^{-1}(G) \subseteq f^{-1}(H)$

Let $a \in G$
So $a \in H$ since $G \subseteq H$
We see that since $a \in G \Rightarrow f^{-1}(a) = y$ for some $y \in f^{-1}(G)$
But also since $a \in F \Rightarrow f^{-1}(a) = y$ for some $y \in f^{-1}(H)$
Thus $f^{-1}(G) \subseteq f^{-1}(H)$ ■

b) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

First we show that $f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H)$
Suppose $a \in f^{-1}(G \cap H)$
Thus we see that $f(a) \in G \cap H$
So, $f(a) \in G$ and $f(a) \in H$
$\Rightarrow a \in f^{-1}(G) \wedge a \in f^{-1}(H)$
$\Rightarrow a \in f^{-1}(G) \cap f^{-1}(H)$

Therefore $f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H)$

Now we show that $f^{-1}(G \cap H) \supseteq f^{-1}(G) \cap f^{-1}(H)$
Let $a \in f^{-1}(G) \cap f^{-1}(H)$
$\Rightarrow a \in f^{-1}(G)$ and $a \in f^{-1}(H)$
$\Rightarrow f(a) \in G$ and $f(a) \in H$
$\Rightarrow f(a) \in G \cap H$
$\Rightarrow a \in f^{-1}(G \cap H)$
Thus $f^{-1}(G \cap H) \supseteq f^{-1}(G) \cap f^{-1}(H)$

So, we see that $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$ ■

c) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

First we show that $f^{-1}(G \cup H) \subseteq f^{-1}(G) \cup f^{-1}(H)$
Suppose $a \in f^{-1}(G \cup H)$
$\Rightarrow f(a) \in G \cup H$
$\Rightarrow f(a) \in G$ or $f(a) \in H$
If $f(a) \in G$, then $a \in f^{-1}(G)$
Now if $f(a) \in H$, then $a \in f^{-1}(H)$
$\Rightarrow a \in f^{-1}(G)$ or $a \in f^{-1}(H)$
$\Rightarrow a \in f^{-1}(G) \cup f^{-1}(H)$
Therefore $f^{-1}(G \cup H) \subseteq f^{-1}(G) \cup f^{-1}(H)$

Now we show that $f^{-1}(G \cup H) \supseteq f^{-1}(G) \cup f^{-1}(H)$
Let $a \in f^{-1}(G) \cup f^{-1}(H)$
$\Rightarrow a \in f^{-1}(G)$ or $a \in f^{-1}(H)$
$\Rightarrow f(a) \in G$ or $f(a) \in H$
$\Rightarrow f(a) \in G \cup H$
$\Rightarrow a \in f^{-1}(G \cup H)$
Thus $f^{-1}(G \cup H) \supseteq f^{-1}(G) \cup f^{-1}(H)$

So, we see that $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ ■

d) $f^{-1}(G - H) = f^{-1}(G) - f^{-1}(H)$

First we show that $f^{-1}(G - H) \subseteq f^{-1}(G) - f^{-1}(H)$
Suppose $a \in f^{-1}(G - H)$
$\Rightarrow f(a) = G - H$
$\Rightarrow f(a) \in G \wedge f(a) \notin H$
$\Rightarrow a \in f^{-1}(G) \wedge a \notin f^{-1}(H)$
$\Rightarrow a \in f^{-1}(G) - f^{-1}(H)$
Therefore $f^{-1}(G - H) \subseteq f^{-1}(G) - f^{-1}(H)$

Next we show that $f^{-1}(G - H) \supseteq f^{-1}(G) - f^{-1}(H)$
Suppose $a \in f^{-1}(G) - f^{-1}(H)$
$\Rightarrow a \in f^{-1}(G) \wedge a \notin f^{-1}(H)$
$\Rightarrow f(a) \in G \wedge f(a) \notin H$
$\Rightarrow f(a) \in (G - H)$
$\Rightarrow a \in f^{-1}(G - H)$
Therefore $f^{-1}(G - H) \supseteq f^{-1}(G) - f^{-1}(H)$

Thus $f^{-1}(G - H) = f^{-1}(G) - f^{-1}(H)$ ■

Problem 15. Write the negation of the convergence definition.

Negation:

$\exists \epsilon > 0, \forall N \in \mathbb{N}$ such that $\exists n \in \mathbb{N}$, choose $n > N$ such that:

$$|a_n - l| \geq \epsilon$$

Write out the meanings:
A sequence is indexed by natural numbers. If a sequence is not convergent, we can find a number that is greater than 0 (ϵ) that is greater than or equal to the difference (absolute difference) between the nth number in a sequence and the supposed limit, l (or any other number). In other words, there does not exist a limit (number) such that a sequence converges to it. So there exists a number in the sequence such that $|a_n - l|$ is greater than or equal to ϵ (where ϵ is greater than 0).

Problem 16. Prove that the sequence $\{\frac{1}{2n}\}$ converges to 0.

We need to show:
$\forall \epsilon > 0, \exists N \in \mathbb{N}$, choose $n > N$ such that

$$\left|\frac{1}{2n} - 0\right| < \epsilon$$

Now, working it out, we see that: $\frac{1}{2n} < \epsilon$
Thus $N > \frac{1}{2\epsilon}$
Thus, choose N to satisfy $N > \frac{1}{2\epsilon}$
It follows that for $n \geq N \Rightarrow \frac{1}{2n} < \epsilon$
Thus, $\frac{1}{2n}$ converges to 0 ■

Problem 17. Let $f : X \to \mathbb{R}$, f is continuous at a if and only if $\forall \epsilon > 0, \exists \delta > 0$, such that $|x - a| < \delta \Rightarrow$

$$|f(x) - f(a)| < \epsilon$$

Prove $f : [1, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x-1}$ is continuous at $x = 10$

Rough work:
We want to show that $|x - 10| < \delta \Rightarrow \left|\sqrt{x-1} - 3\right| < \epsilon$

$$\Rightarrow \left|\sqrt{x-1} - 3\right| < \epsilon$$

$$\Rightarrow -\epsilon < \sqrt{x-1} - 3 < \epsilon$$

$$\Rightarrow -\epsilon + 3 < \sqrt{x-1} < \epsilon + 3$$

$$\Rightarrow (3-\epsilon)^2 < x-1 < (3+\epsilon)^2$$

$$\Rightarrow (3-\epsilon)^2 - 9 < x-1-9 < (3+\epsilon)^2 - 9$$

$$\Rightarrow (3-\epsilon)^2 - 9 < x-10 < (3+\epsilon)^2 - 9$$

Let $\epsilon > 0$, choose $\delta = \max\{((3-\epsilon)^2 - 9), ((3+\epsilon)^2 - 9)\}$. Then $|x-10| < \delta \Rightarrow$

$$\left|\sqrt{x-1} - 3\right| < \epsilon$$

So f is continuous at $x = 10$ ■

Problem 18. Let f be defined as $f(x) = x^2$. Determine $f'(3)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

We want to prove $f'(3) = 2(3) = 6$
For constructing an $\epsilon - \delta$ proof: we need to find an appropriate δ

$$\left|\frac{x^2-9}{x-3} - 6\right|$$

$$= \left|\frac{(x+3)(x-3)}{(x-3)} - 6\right|$$

$$= |(x+3) - 6|$$

$$= |x-3|$$

So we let $\delta = \epsilon$

$\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x-3| < \delta \Rightarrow$

$$\left|\frac{f(x)-f(3)}{x-3} - f'(3)\right| < \epsilon$$

$$\Rightarrow \left|\frac{(x+3)(x-3)}{(x-3)} - 6\right| < \epsilon$$

$$\Rightarrow |(x+3) - 6| < \epsilon$$

$$\Rightarrow |x-3| < \epsilon$$

Thus $f'(3) = 2(3) = 6$ ■