Assignment 2 - Math Camp

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Problem 1. What are the dimensions of the following subsets of $\mathbb{R}^3$?

1. The origin:
   The origin ($\mathbf{0}$) has 0 dimensions.

2. A line through the origin:
   As a line through the origin can be modeled using one vector (the basis is equal to 1) in $\mathbb{R}^3$ with the vector being of form:
   $$\mathbf{v}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$
   where $a, b, c \in \mathbb{R}$.
   Thus, $dim(\mathbf{v}_1) = 1$

3. A plane which passes through the origin:
   As a plane which passes through the origin can be modeled using two vectors (the basis is equal to 2) in $\mathbb{R}^3$, with the matrix (of vectors) being of form:
   $$A\mathbf{c} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
   $$\mathbf{v}_1 = \begin{pmatrix} t \\ u \\ v \end{pmatrix}$$
   $$\mathbf{v}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
   where $t, u, v, x, y, z \in \mathbb{R}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
   Thus, $dim(A) = 2$.

Problem 2. For $\mathbf{a}, \mathbf{x} \in \mathbb{R}^n$, consider the equation $\mathbf{a} \cdot \mathbf{x} = 0$ and its solution set $X(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = 0\}$. Show:

1. that X($\mathbf{a}$) is a linear subspace
   We need to show that the three properties of linear subspaces are satisfied:

   (a) Show $\mathbf{0} \in X(\mathbf{a})$

   $\mathbf{a} \cdot \mathbf{x} = a_1x_1 + a_2x_2 + ... + a_nx_n$
   $X(\mathbf{a}) = \mathbf{0} \Rightarrow a_10 + a_20 + ... + a_n0 = 0$
   The $\mathbf{a} \cdot \mathbf{x} = 0$ constraint holds for $X(\mathbf{a}) = 0$, thus $\mathbf{0} \in X(\mathbf{a})$

(b) Show $X(\mathbf{a})$ is closed under addition

Let $\mathbf{x}_b, \mathbf{x}_c \in X(a)$
Thus, $\mathbf{a} \cdot \mathbf{x}_b = 0$ and $\mathbf{a} \cdot \mathbf{x}_c = 0$
Now, we see:

$$\begin{gathered}\mathbf{a} \cdot (\mathbf{x}_b + \mathbf{x}_c) \\ = a_1(x_{b1} + x_{c1}) + a_2(x_{b2} + x_{c2}) + ... + a_n(x_{bn} + x_{cn}) \\ = a_1x_{b1} + a_1x_{c1} + a_2x_{b2} + a_2x_{c2} + ... + a_nx_{bn} + a_nx_{cn} \\ = a_1x_{b1} + a_2x_{b2} + ... + a_nx_{bn} + a_1x_{c1} + a_2x_{c2} ... + a_nx_{cn} \\ = \mathbf{a} \cdot \mathbf{x}_b + \mathbf{a} \cdot \mathbf{x}_c \\ = 0 + 0\end{gathered}$$

Thus $X(\mathbf{a})$ is closed under addition.

(c) Show $X(\mathbf{a})$ is closed under scalar multiplication

Let $\mathbf{x} \in X(\mathbf{a})$
Thus $\mathbf{a} \cdot \mathbf{x} = 0$
Now let $c \in \mathbb{R}$
We see that

$$\begin{gathered}ca_1x_1 + ca_2x_2 + ... + ca_nx_n \\ = c(a_1x_1 + a_2x_2 + ... + a_nx_n) \\ = c\mathbf{a} \cdot \mathbf{x} \\ = c(0) \\ = 0\end{gathered}$$

Thus $(c\mathbf{a}) \cdot \mathbf{x} = 0$

2. the dimension of $X(\mathbf{a})$

We see that $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$
Solving for x_1: $x_1 = -\frac{a_2}{a_1}x_2 + ... + -\frac{a_n}{a_1}x_n$
Assuming that $a_i \in \mathbb{R} - \{0\} \quad \forall i \in \{1, 2, ..., n\}$.
Thus:

$$\text{X}(\mathbf{a}) = \begin{pmatrix} -\frac{a_2}{a_1} \\ 1 \\ 0 \\ ... \\ 0 \\ 0 \end{pmatrix} \text{x}_2 + \begin{pmatrix} -\frac{a_3}{a_1} \\ 0 \\ 1 \\ ... \\ 0 \\ 0 \end{pmatrix} x_3 + ... + \begin{pmatrix} -\frac{a_n}{a_1} \\ 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix} x_n$$

$$\text{X}(\mathbf{a}) = \text{span}\left\{ \begin{pmatrix} -\frac{a_2}{a_1} \\ 1 \\ 0 \\ ... \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{a_3}{a_1} \\ 0 \\ 1 \\ ... \\ 0 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} -\frac{a_n}{a_1} \\ 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix} \right\}$$

Since the $(n-1)$ vectors in $X(\mathbf{a})$ are linearly independent, then $dim(X(\mathbf{a})) = n - 1$

Problem 3. Suppose $\mathbf{a}, \mathbf{x}, \mathbf{y} \in R^n$ and $\mathbf{a} \cdot \mathbf{x} > \mathbf{a} \cdot \mathbf{y}$. Does it follow that $\mathbf{x} > \mathbf{y}$? [Hint: do not divide both sides by $\mathbf{a}$]

No, it does not follow that $\mathbf{x} > \mathbf{y}$.
Consider the following counterexample:

Let:

$$\mathbf{a} = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$
$$\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

We see that $\mathbf{a} \cdot \mathbf{x} > \mathbf{a} \cdot \mathbf{y}$ as:
$\mathbf{a} \cdot \mathbf{x} = 2(3) + 2(2) + 1(1) = 11$
$\mathbf{a} \cdot \mathbf{y} = 2(1) + 2(2) + 1(3) = 9$

However, we see in this case that $\mathbf{x} \not> \mathbf{y}$

Thus $\mathbf{a} \cdot \mathbf{x} > \mathbf{a} \cdot \mathbf{y}$ does not imply $\mathbf{x} > \mathbf{y}$

Problem 4. Let X be a vector space and consider a function $f : X \to \mathbb{R}$ defined for some $\mathbf{a} \in X$ defined as $f_a(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$.

Note, I made the assumption that $X = \mathbb{R}^n$

(i) Prove that $f_a(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ is a linear function.

In order to prove $f_a(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ is a linear function, we need to show that: (a) $f_a(\mathbf{x}+\mathbf{y}) = f_a(\mathbf{x}) + f_a(\mathbf{y})$, and (b) $f_a(r\mathbf{x}) = r f_a(\mathbf{x})$

(a) Show $f_a(\mathbf{x}+\mathbf{y}) = f_a(\mathbf{x}) + f_a(\mathbf{y})$

Let $\mathbf{x}, \mathbf{y} \in X$
Since X is a vector space and $\mathbf{x}, \mathbf{y} \in X$, thus $(\mathbf{x}+\mathbf{y}) \in X$
Observe that:

$$\begin{aligned} f_a(\mathbf{x}+\mathbf{y}) &= \mathbf{a} \cdot (\mathbf{x}+\mathbf{y}) \\ &= \mathbf{ax} + \mathbf{ay} \\ &= f_a(\mathbf{x}) + f_a(\mathbf{y}) \\ \Rightarrow f_a(\mathbf{x}+\mathbf{y}) &= f_a(\mathbf{x}) + f_a(\mathbf{y}) \end{aligned}$$

(b) Show $f_a(r\mathbf{x}) = r f_a(\mathbf{x})$

Let $\mathbf{x} \in X$
Since X is a vector space, it is closed under scalar multiplication, and thus $rx \in X$
Observe that:

$$f_a(r\mathbf{x}) = \mathbf{a} \cdot (r\mathbf{x})$$

$$= \mathbf{a} \cdot (rx_1 + rx_2 + ... + rx_n)$$
$$= a_1 rx_1 + a_2 rx_2 + ... + a_n rx_n$$
$$= ra_1 x_1 + ra_2 x_2 + ... + ra_n x_n$$
$$= r(a_1 x_1 + a_2 x_2 + ... + a_n x_n)$$
$$= r(\mathbf{a} \cdot x)$$
$$= rf_a(\mathbf{x})$$
$$\Rightarrow f_a(r\mathbf{x}) = rf_a(\mathbf{x})$$

Thus, $f_a(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ is a linear function

(ii) Let $X^* = \{f : X \to \mathbb{R} : f \text{ is linear}\}$ be the set of all linear functions from X into $\mathbb{R}$. Prove that for all $f \in X^*$ there exists an $\mathbf{a} \in X$ such that $f(x) = \mathbf{a} \cdot \mathbf{x}$.

We see that if $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, then f is continuous and linear.
Assume f is linear, and let $\mathbf{e}_1 = (1, 0, ..., 0)^T, \mathbf{e}_2 = (0, 1, ..., 0)^T, ..., \mathbf{e}_n = (0, 0, ..., 1)^T$
We see that every $\mathbf{x} \in X$ has a unique representation of the form:

$$\mathbf{x} = \sum_{i=1}^{n} x_i e_i$$

where $x_i \in \mathbb{R}$
Thus $f(\mathbf{x}) = \sum_{i=1}^{n} x_i f(e_i)$
Now, if $\mathbf{a}$ is the vector such that its ith component: $a_i = f(\mathbf{e}_i)$
Then $f(\mathbf{x}) = \sum_{i=1}^{n} x_i a_i = \mathbf{a} \cdot \mathbf{x}$

(iii) Define function addition as $f + g = f(\mathbf{x}) + g(\mathbf{x})$ and function scaling as $\alpha f = \alpha f(\mathbf{x})$ over the set X^*. Prove or disprove the following statement: X^* with the defined operations is a linear vector space.

Let $f, g \in X^*$
We define $f + g = f(\mathbf{x}) + g(\mathbf{x})$, and let $f(\mathbf{x}) = \mathbf{ax}$ and $g(\mathbf{x}) = \mathbf{bx}$
Since f and g are linear, and $\mathbf{ax}, \mathbf{bx} \in X^*$, it follows that $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{x} \in X^*$. Thus:

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{x} = (a_1 + b_1)x_1 + (a_2 + b_2)x_2 + ... + (a_n + b_n)x_n$$
$$= a_1 x_1 + b_1 x_1 + a_2 x_2 + b_2 x_2 + ... + a_n x_n + b_n x_n$$
$$= (a_1 x_1 + a_2 x_2 + ... + a_n x_n) + (b_1 x_1 + b_2 x_2 + ... + b_n x_n)$$
$$= \mathbf{ax} + \mathbf{bx}$$
$$= f(\mathbf{x}) + g(\mathbf{x})$$
$$= f + g$$

Therefore $f + g \in X^*$. So X^* is closed under addition.

Now, we define $\alpha f = \alpha f(\mathbf{x})$ where $\alpha \in \mathbb{R}$ and $f \in X^*$.
Since f is linear, and $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} \in X^*$, we see $\mathbf{a} \cdot (\alpha \mathbf{x}) \in X^*$, and observe that:

$$\mathbf{a} \cdot (\alpha \mathbf{x}) = \alpha a_1 x_1 + \alpha a_2 x_2 + .. + \alpha a_n x_n$$
$$= \alpha(a_1 x_1 + a_2 x_2 + .. + a_n x_n)$$
$$= \alpha(\mathbf{a}\mathrm{x})$$

$$= \alpha f(\mathbf{x})$$
$$= \alpha f$$

So $\alpha f \in X^*$. Thus X^* is closed under scalar multiplication.

Because of linearity, $\mathbf{0} \in X^*$

Since $\mathbf{0} \in X^*$, and $X*$ is closed under addition and scalar multiplication, X^* is a vector space.[⋆]

(iv) What is dimension of X^*?

We want to show that $f_1, ..., f_n$ form a basis for X^*. To show this basis, we need to find the span and show linear independence.
As before, we let $f(\mathbf{e}_i) = a_i \quad \forall i \in \{1, 2, ..., n\}$
Also, let A be an $n \times n$ matrix with the rows: $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$
So, we see that $A\mathbf{x} = \mathbf{a}$
Since $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ are linearly independent, we see that A has full rank.
Now, we can express f as $f = a_1 f_1 + a_2 f_2 + ... + a_n f_n$

$$\Rightarrow f(\mathbf{x}) = a_1 f_1(\mathbf{x}) + a_2 f_2(\mathbf{x}) + ... + a_n f_n(\mathbf{x})$$
$$= a_1 x_1 + a_2 x_2 + ... + a_n x_n$$
$$= \mathbf{a}^T \mathbf{x}$$

So we see that $f_1, f_2, ..., f_n$ span X^*

Now we show linear independence
Suppose $c_1 f_1 + c_2 f_2 + ... + c_n f_n = 0$ for some $c_i \in \mathbb{R}$ where $i \in \{1, 2, ..., n\}$
$\Rightarrow c_1 f_1(\mathbf{x}) + c_2 f_2(\mathbf{x}) + ... + c_n f_n(\mathbf{x}) = 0$
$\Rightarrow c_1 f_1(\mathbf{e}_1) + c_2 f_2(\mathbf{e}_2) + ... + c_n f_n(\mathbf{e}_n) = 0$
We see it is the case that $c_1 = 0, c_2 = 0, ..., c_n = 0$.
Therefore $c_1 f_1(\mathbf{e}_1) + c_2 f_2(\mathbf{e}_2) + ... + c_n f_n(\mathbf{e}_n) = 0 \Rightarrow c_1 = 0,\ c_2 = 0, ..., c_n = 0$
This implies linear independence.

Hence $f_1, ..., f_n$ form a basis for X^*. So we see $dim(X^*) = dim(X) = n$

Problem 5. Prove that the set Z is a subspace of $\mathbb{R}^3$.

$$Z = \{[x_1, x_2, x_3] : 4x_1 - x_2 + 5x_3 = 0\}$$

We need to show that the three properties of linear subspaces are satisfied:[*]

(a) Show $\mathbf{0} \in Z$

We see that for $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
Inputting this into $4x_1 - x_2 + 5x_3$, we get $4(0) - (0) + 5(0) = 0$
Thus $\mathbf{0} \in Z$

[⋆]As $X = \mathbb{R}^n$, we only need to check that X is closed under addition, closed under scalar multiplication, and that the zero vector is in X (Simon and Blume Thm 27.1).

(b) Show Z is closed under addition

Let $[x_1, x_2, x_3], [y_1, y_2, y_3] \in Z$
Thus $4x_1 - x_2 + 5x_3 = 0$ and $4y_1 - y_2 + 5y_3 = 0$
Adding these two vectors, we get:

$$4(x_1 + y_1) - (x_2 + y_2) + 5(x_3 + y_3)$$
$$= 4x_1 + 4y_1 - x_2 - y_2 + 5x_3 + 5y_3$$
$$= 4x_1 - x_2 + 5x_3 + 4y_1 - y_2 + 5y_3$$
$$= 0 + 0$$

Thus Z is closed under addition.

(c) Show Z is closed under scalar multiplication

Let $[x_1, x_2, x_3], \in Z$
Thus, $(4x_1 - x_2 + 5x_3) = 0$
We are required to show that $4cx_1 - cx_2 + 5cx_3 = 0$ where $c \in \mathbb{R}$
We see:

$$4cx_1 - cx_2 + 5cx_3$$
$$= c(4x_1 - x_2 + 5x_3)$$
$$= c(0)$$
$$= 0$$

Since $4cx_1 - cx_2 + 5cx_3 = 0$, Z is closed under scalar multiplication.

Thus Z is a linear subspace.

Problem 6. Determine the null space and range of each of the following linear operators on $\mathbb{R}^3$.

(i) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$

To find the null space of $L(\mathbf{x})$, we need to find $\mathbf{x} : L(\mathbf{x}) = \mathbf{0}$
$N(L) = \{(x_1, x_2, x_3) : (x_3, x_2, x_1)^T = \mathbf{0}\}$
Thus $\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
So, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$
Therefore $N(L) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$
Thus the nullity of L is 0. Using the rank-nullity theorem, we see that $rank(L) = dim(Col(L)) = 3$
So we see that:
$range(L) = \mathbb{R}^3$

(ii) $L(x) = (x_1, x_1, x_1)^T$

To find the null space of $L(\mathbf{x})$, we need to find $\mathbf{x} : L(\mathbf{x}) = 0$

$N(L) = \{(x_1, x_2, x_3) : (x_1, x_1, x_1)^T = \mathbf{0}\}$

Thus $\begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Thus, $x_1 = 0$ and $x_2, x_3 \in \mathbb{R}$

Thus $N(L) = \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$

Thus a basis for the $N(L)$ is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$range(L) = \left\{ x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$

So basis for $range(L)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

This is consistent with what we find using the rank-nullity theorem. Since nullity of L is 2, then $dim(Col(L)) = 1$. This implies our range is of 1 dimension.

Problem 7. What are the range of angles between vectors in $x \in H_a^{\leq}(0)$ and the vector a?
The range of angles between vectors in $x \in H_a^{\leq}(0)$ and the vector a is between 90 and 270 degrees.

What are the ranges of the angles between vectors $x \in H_a^{\geq}(0)$ and the vector a?
The range of angles are also between vectors in $x \in H_a^{\geq}(0)$ and the vector a is between 0 and 90, and between 270 and 360 degrees.

Look at the appendix to see illustrations of half-spaces in $\mathbb{R}^2$.

Problem 8. Which of the following are subspaces of the vector space $M_{2,2}$ of 2×2 matrices? Justify your answer.

(i) the set of 2×2 real symmetric matrices.

A real diagonal matrix has the form of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

where $a, b, c \in \mathbb{R}$

(a) Show $\mathbf{0}$ is in the set

If $a = b = c = 0$, then $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) Show closure under addition

Let A$=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

where $a, b, c \in \mathbb{R}$

and B$=\begin{pmatrix} e & f \\ f & g \end{pmatrix}$

where $e, f, g \in \mathbb{R}$

We see that $A + B = \begin{pmatrix} a+e & b+f \\ b+f & c+g \end{pmatrix}$

Since $(a+e),(b+f),(c+g) \in \mathbb{R}$, the set is closed under addition

(c) Show closure of scalar multiplication

Let A=$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$
where $a, b, c \in \mathbb{R}$
We see that $rA = r\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} ra & rb \\ rb & rc \end{pmatrix}$
Since $ra, rb, rc \in \mathbb{R}$, then the set is closed under scalar multiplication.

Since these three properties are satisfied, the set of the set of 2×2 real symmetric matrices is a subspace.

(ii) the set of 2×2 real diagonal matrices.
A real diagonal matrix has the form of $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
where $a, b \in \mathbb{R}$

(a) Show $\mathbf{0}$ is in the set
Obviously, if $a = b = 0$, then $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mathbf{0}$

(b) Show closure under addition

Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
where $a, b \in \mathbb{R}$
and $B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$
where $c, d \in \mathbb{R}$

Thus, $A + B = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix}$
So the set is closed under addition since $(a+c),(b+d) \in \mathbb{R}$

(c) Show closure of scalar multiplication
Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
where $a, b \in \mathbb{R}$
We see that $rA = r\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ra & 0 \\ 0 & rb \end{pmatrix}$
Since $ra, rb \in \mathbb{R}$, the set is closed under scalar multiplication

Since these three properties are satisfied, the set of the set of 2×2 real diagonal matrices is a subspace.

(iii) the set of 2×2 real "singular" matrices (remember M is singular if $det(M) = 0$).

No. Consider the following matrices:
Let $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$
where $a \in \mathbb{R} - \{0\}$
and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$
where $b \in \mathbb{R} - \{0\}$

Obviously, $A \in M_{2\times2}$ and $B \in M_{2\times2}$

If the set of all 2×2 singular matrices were a subspace of $M_{2\times 2}$, we would need them to be closed under addition.
We see however, that $A + B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
This is not contained in our set, as $ab - 0 \neq 0$
Thus the set of all 2×2 singular matrices of $M_{2\times 2}$ is not a subspace.

(iv) the zero matrix.

(a) Show $\mathbf{0}$ is in the set
This is obviously satisfied.

(b) Show closure under addition
We see that $\mathbf{0} + \mathbf{0} = \mathbf{0}$
So $\mathbf{0}$ is closed under addition

(c) Show closure of scalar multiplication
We see that $r\mathbf{0} = \begin{pmatrix} r0 & r0 \\ r0 & r0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Thus $\mathbf{0}$ is closed under scalar multiplication

So the zero matrix is a subspace.

(v) the set of all 2×2 nonsingular matrices

No. Consider the following matrices:
Let $A = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix}$
Observe that $det(A) = 2(1) - 6(1) \neq 0$
and $B = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$
Observe $det(B) = 1(2) - 0(3) \neq 0$

We see that $A + B = \begin{pmatrix} 3 & 1 \\ 9 & 3 \end{pmatrix}$
Notice that $det(A + B) = 3(3) - 9(1) = 0$
Thus the set of all 2×2 nonsingular matrices is not closed under scalar multiplication
Thus it is a not a subspace.

Problem 9. Let $f : R^n \to R^m$ be a linear function such that for all $\mathbf{y} \in R^m$ the set $\{\mathbf{x} \in R^n : f(\mathbf{x}) = \mathbf{y}\}$ is a singleton.

Proposition 1: An invertible matrix has to be square (i.e. have dimensions $n \times n$)

By definition, A^{-1} is the inverse of A iff $AA^{-1} = A^{-1}A = I^\star$
Let A be the matrix represented by f. Thus A is $n \times m$
We will first show that $n = m$
Looking at $AA^{-1} = I_n$, we see that A^{-1} has to have the dimensions of $m \times n$, as our Identity matrix has to be $n \times n$.
Now using this, consider $A^{-1}A = I_m$, where I_m is an $m \times m$ identity matrix.
Since by definition, $AA^{-1} = A^{-1}A$, we see that $I_n = I_m$.
Thus $m = n$, in others words, A and A^{-1} matrices both have dimensions of $n \times n$.

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*The inverse of a matrix is unique. To show this, assume the matrix, A has two inverses: B and C. Thus $AB = BA = I_n$, and $AC = CA + I_n$ by definition of an inverse matrix. Thus we see that $C = CI = C(AB) = (CA)B = IB = B$, so $C = B$, therefore the inverse matrix of an invertible matrix is unique.

(i) Is the linear function f invertible?

If f is invertible, it means that it is one-to-one and onto
Thus, we are required to prove that f is one-to-one and onto
To prove that f is one-to-one, we would need to show that each $y \in \mathbb{R}^m$ is the image of at most one $x \in \mathbb{R}$.
As the set $\{x \in \mathbb{R}^n : f(x) = y\}$ is a singleton for each y, it follows that f is one-to-one.
To prove that f is onto, we would need to show that each $y \in \mathbb{R}^m$ is the image of at least one $x \in \mathbb{R}^n$
As the set $\{\mathbb{R}^n : f(x) = y\}$ holds for each $y \in \mathbb{R}^m$, it follows that f is onto.

Since f is onto and one-to-one, it follows that f is invertible.

(ii) If f can be represented by matrix A, then show A is invertible if and only if $rank(A) = m = n$.

$\Leftarrow$) As invertibility means one-to-one and onto, we are required to prove these two properties are satisfied.
Suppose $rank(A) = n = m$
By rank-nullity theorem, we know that $rank(A) + dim(N(A)) = n$.
Since $rank(A) = n$, it follows that $dim(N(A)) = 0$
$dim(N(A)) = 0 \Rightarrow f$ is one-to-one.*

Now we need to show that f is onto.
Thus we need to show that $f(x) = b$ has solutions for every $b \in \mathbb{R}^m$
Since $rank(A) = dim(Im(A))$, then $dim(Im(A)) = n$.
The definition of $Im(A) = \{b \in \mathbb{R}^m : Ax = b$ has at least one solution$\}$
$\Rightarrow Im(A) = \{A(x) : x \in \mathbb{R}^n\}$
Thus it follows that f is onto
Since these two properties are satisfied, A is invertible.

$\Rightarrow$) Suppose A is invertible.
We are required to show that $rank(A) = n = m$
By proposition 1, A is a square matrix of $n \times n$ dimensions (since m=n).
From Rank-nullity theorem, we know that $rank(A) + dim(N(A)) = n$
If we show that $dim(N(A) = 0$, it will immediately follow that $rank(A) = n = m$.
Consider $Ax = 0$
Since A is invertible, we know that AA^{-1} holds, where is A^{-1} is the inverse matrix of A. Thus:

$$A^{-1}A = I_n$$

Now consider $Ax = 0$. We see that:

$$Ax = 0$$
$$\Rightarrow A^{-1}Ax = A^{-1}0$$
$$\Rightarrow A^{-1}0 = 0$$
$$\Rightarrow I_n x = 0$$

Since I_n has the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

*Suppose $f(x_1) = f(x_2)$. Thus $\mathbf{0} = f(x_1) - f(x_2) = f(x_1 - x_2)$ by linear transformation. Thus $f(x_1 - x_2) = 0$. This implies that $x_1 - x_2 = \mathbf{0}$, or $x_1 = x_2$. Thus f is one-to-one.

Thus it follows that $x = \mathbf{0}$ and that is the only solution it can have
Thus, the dimension of the null space is 0
Using the rank-nullity theorem, we see: $rank(A) + 0 = n$, thus $rank(A) = n = m$.

(iii) Suppose that an $n \times n$ matrix A is invertible. Does it follow that $[Ax = 0] \Rightarrow [x = 0]$?

Suppose that A is an $n \times n$ invertible matrix.
Since A is invertible, we know that AA^{-1} holds, where is A^{-1} is the inverse matrix of A. Thus:

$$A^{-1}A = I_n$$

Now consider $Ax = 0$. We see that:

$$\begin{aligned} Ax &= 0 \\ \Rightarrow A^{-1}Ax &= A^{-1}0 \\ \Rightarrow A^{-1}0 &= 0 \\ \Rightarrow I_n x &= 0 \end{aligned}$$

Since I_n has the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Thus it follows that $x = \mathbf{0}$

(iv) Suppose that A is an $n \times n$ matrix and that $[Ax = 0] \Rightarrow [x = 0]$. Does it follow that A is invertible?

Suppose that A is an $n \times n$ matrix, and that $[Ax = 0] \Rightarrow [x = 0]$.
Thus, there are no free variables.
Therefore, A must have n pivot columns.
Since A is $n \times n$, the pivots in echelon form of A must be on the main diagonal
Hence, A is row equivalent to I_n, $(A \sim I_n)$
We see that since each step of the row reduction of A corresponds to the left-multiplication of an elementary matrix, we see there exist elementary matrices $E_1, ..., E_s$ such that:

$$A \sim E_s A \sim E_1(E_2 A) \sim ... \sim E_s(E_{s-1}...E_1 A) = I_n$$
$$E_s...E_1 A = I_n$$

Since the product of $E_s...E_1$ of invertible matrices is invertible, we see $(E_s...E_1)$ is invertible.*
Thus:

$$(E_s...E_1)^{-1}(E_s...E_1)A = (E_s...E_1)^{-1}I_n$$
$$A = (E_s...E_1)^{-1}$$

As A is the inverse of an invertible matrix, A is invertible and has the form:

$$A^{-1} = [(E_s...E_1)^{-1}]^{-1} = (E_s...E_1)$$

Thus it does follow that A is invertible.

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*We know that each elementary matrix (which represents a certain row operation) has an inverse elementary matrix (as the row operation can be reversed).
Now we show that the product of invertible matrices is an invertible matrix. Let A and B be two invertible matrices. Thus $\det(B) \neq 0$ and $\det(A) \neq 0$. Using properties of determinants, we see that $\det(A) \cdot \det(B) \neq 0 \Rightarrow \det(AB) \neq 0$. Thus AB is an invertible matrix. We can use induction to show this holds for the product of n matrices.

Problem 10. Let $\mathbf{y} \in \mathbb{R}^n$ be a netput vector where each element y_i for $i = 1, ..., n$ is a commodity. If $y_i < 0$ then y_i is an input into a production process. If $y_i > 0$ then y_i is an output of the production process. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a transformation function and we define the set $Y = \{\mathbf{y} \in \mathbb{R}^n : F(\mathbf{y}) \leq 0\}$ and we'll call Y a technology. The technology Y describes all the feasible production plans a firm can choose (i.e., combinations of feasible inputs and outputs). We will assume that Y is a convex set. Now let $\mathbf{p} \in \mathbb{R}^n$ where $\mathbf{p} >> 0$ be vector prices for the n commodities in any bundle $\mathbf{y} \in Y$. Note that for fixed $\mathbf{p}$, we can define a function $T_p(\mathbf{y}) = \mathbf{p} \cdot \mathbf{y}$ for all $\mathbf{y} \in Y$. The function $T_p(\mathbf{y})$ represents the profit (revenue minus costs) of choosing production plan $\mathbf{y}$ when input and output prices are given by $\mathbf{p}$. Firms want to choose some feasible production plan $\mathbf{y}^*$ that satisfies $T_p(\mathbf{y}^*) = \max\{\mathbf{p} \cdot \mathbf{y} : \mathbf{y} \in Y\} = \pi(\mathbf{p})$.

(i) If $\pi(\mathbf{p})$ is the maximum achievable profit for firms given technology Y and prices $\mathbf{p}$ the profit-maximizing production plans are $\{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$. Now consider the set $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$. Is this a hyperplane? If so, show that Y is contained in one of the half-spaces of $\mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})$ and specify which one (i.e., upper or lower).

$\{\mathbf{y} \in R^n : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$ is obviously a hyperplane as it has the form $\{\mathbf{x} \in \mathbb{R}^L : \mathbf{p} \cdot \mathbf{x} = c\}$ (as seen in MWG p. 64). $\mathbf{p}$ is the normal vector to this hyperplane.

Now we show that $\{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\}$ is the half space that contains Y. Let $\mathbf{y}^* \in \{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$. Thus by definition of $\pi(\mathbf{p})$: $\mathbf{p} \cdot \mathbf{y}^* \geq \mathbf{p} \cdot \mathbf{y} \quad \forall \mathbf{y} \in Y$ (so obviously, $\nexists\, \mathbf{y} \in Y$ such that $\mathbf{p} \cdot \mathbf{y} > \mathbf{p} \cdot \mathbf{y}^*$)
Thus $Y \subseteq \{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\}$

(ii) Does the set $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$ "touch" the technology Y? i.e, is it the case that

$$\min_{\mathbf{y} \in Y} |\mathbf{p} \cdot \mathbf{y} - \pi(\mathbf{p})| = 0$$

Yes, it is the case that the set $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$ touches the technology set, Y.
The $\pi(\mathbf{p})$ set is defined as $\pi(\mathbf{p}) = \max\{\mathbf{p} \cdot \mathbf{y} : \mathbf{y} \in Y\}$, where $Y \subseteq \mathbb{R}^n$
Since $\mathbf{y}^*$ is the $\mathbf{y} \in Y$ that achieves the max profit, i.e. $\pi(\mathbf{p}) = \mathbf{p} \cdot \mathbf{y}^* \geq \mathbf{p} \cdot \mathbf{y}$, $\forall \mathbf{y} \in Y$, we know that the set $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$ touches the set Y as $\mathbf{y}^* \in Y$

If so, what does this say about the value of $F(\mathbf{y}^*)$ for any $\mathbf{y}^* \in \{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$?

We see that $F(\mathbf{y}^*)$ for any $\mathbf{y}^* \in \{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$ is on the transformation frontier or border of our production set (since $\min_{\mathbf{y} \in Y} |\mathbf{p} \cdot \mathbf{y} - \pi(\mathbf{p})| = 0$), thus $F(\mathbf{y}^*) = 0$ (MWG p. 128).

(iii) (Duality) Consider the sets of the form $A(\mathbf{p}) = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\}$ and we create a collection of sets $\{A(\mathbf{p}) \subseteq \mathbb{R}^n : \mathbf{p} \in \mathbb{R}^n\}$. Show that the following equality is true.

$$Y = \bigcap_{\mathbf{p} \in R^n} A(\mathbf{p})$$

$\subseteq$) First we show that

$$Y \subseteq \bigcap_{\mathbf{p} \in R^n} A(\mathbf{p})$$

Let $\mathbf{y} \in Y$
As showed in part (i), we know that the half-space: $\{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\} \supseteq Y$
Also, since $Y \subseteq \mathbb{R}^n \Rightarrow \{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\} \subseteq \{\mathbf{y} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\} \quad \forall \mathbf{p} \in \mathbb{R}^n$
So, $Y \subseteq \{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\} \subseteq \{\mathbf{y} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p})\} \;\; \forall \mathbf{p} \in \mathbb{R}^n$
Thus $Y \subseteq \bigcap_{\mathbf{p} \in R^n} A(\mathbf{p})$

$\supseteq$) Now we show that

$$Y \supseteq \bigcap_{\mathbf{p} \in R^n} A(\mathbf{p})$$

Suppose by contraposition, that Y is a nonempty, closed and convex set, and that $\mathbf{x} \notin Y$
We can thus apply lemma 1 of Minkowski's separating hyperplane theorem* : $\exists\, \mathbf{p} \in \mathbb{R}^n$ such that:

$$\mathbf{p} \cdot \mathbf{x} > \max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})$$

Thus $\mathbf{x} \notin \bigcap_{\mathbf{p} \in \mathbb{R}^n} A(\mathbf{p})$
This proves $Y \subseteq \bigcap_{\mathbf{p} \in \mathbb{R}^n} A(\mathbf{p})$

*Minkowski's separating hyperplane theorem (lemma 1):
For every non-empty, closed and convex set $X \subseteq \mathbb{R}^n$ and every point $y \in \mathbb{R}^n - X$, there exists a non-null vector $\mathbf{p} \in R^n - \{0\}$ such that $\mathbf{p} \cdot \mathbf{y} > \sup\{\mathbf{p} \cdot \mathbf{x} : \mathbf{x} \in X\}$.

Appendix

Figure 1: Depicted below is the "lower" half space, $H^{\leq}_{\mathbf{a}}(0)$

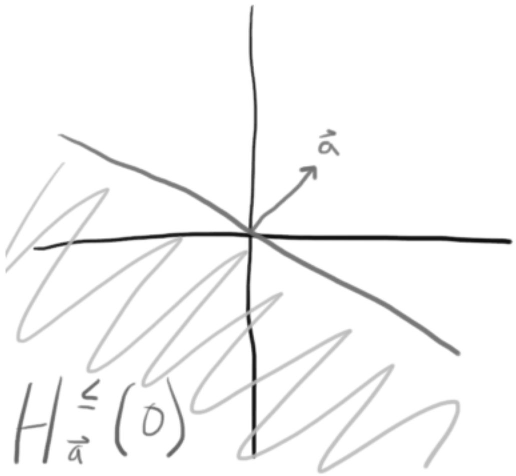

Figure 2: Depicted below is the "upper" half space, $H^{\geq}_{\mathbf{a}}(0)$