Assignment 3 - Math Camp

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Problem 1. Consider an economy with three sectors: farming, manufacturing and textiles. Each sector produces its goods and then trades (through barter) with the other sectors. Hence the three kinds of goods move between sectors. Let the following table describe the results of trade.

	F	M	T
F	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
M	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$
T	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$

The table is interpreted is as follows. Let x_1 be the total value of farm goods, x_2 the total value of manufacturing goods and x_3 the total value of textile goods. The first column of the table describes where the farming sectors output goes: 1/2 to themselves, 1/4 to manufacturing and 1/4 to textiles. The first row describes the value of the farming sectors inputs: 1/2 from farming itself, 1/3 manufacturing goods and 1/2. Hence the total value of farm goods is $x_1 = (1/2)x_1 + (1/3)x_2 + (1/2)x_3$. Doing the same for the other sectors gives the system

$$x_1 = \frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3$$
$$x_2 = \frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3$$
$$x_3 = \frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3$$

Turn the above system into a homogenous system and then solve the system to determine the total values of goods x_1, x_2, x_3.

Putting this into matrix format, we see that:

$$\mathbf{x} = A\mathbf{x}$$

Where

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

Thus, it follows that:

$$\begin{aligned} \mathbf{x} &= A\mathbf{x} \\ \Rightarrow \mathbf{x} - A\mathbf{x} &= \mathbf{0} \\ \Rightarrow (\mathbf{I} - A)\mathbf{x} &= \mathbf{0} \end{aligned}$$

Where

$$I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & -\frac{2}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & -\frac{3}{4} \end{pmatrix}$$

Putting this into augmented matrix, we get:

$$\left(\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{2}{3} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{3} & -\frac{3}{4} & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{3} & -\frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So we get

$$x_1 - \frac{5}{3}x_3 = 0$$
$$x_2 - x_3 = 0$$

where x_3 is a free variable. Now set $x_3 = r$:

Our solution set (of infinite solutions) is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left\{ \begin{pmatrix} \frac{5}{3} \\ 1 \\ 1 \end{pmatrix} r : r \in \mathbb{R} \right\}$$

Problem 2. Determine whether the following system is inconsistent. If not inconsistent and no free variables, find the unique solution. If there are free variables find all solutions (describe the set).

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$2x_1 + 3x_2 - x_3 - x_4 = 2$$
$$3x_1 + 2x_2 + x_3 + x_4 = 5$$
$$3x_1 + 6x_2 - x_3 - x_4 = 4$$

To find solution(s), we make an augmented matrix and reduce the matrix to rref

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & -1 & -1 & 2 \\ 3 & 2 & 1 & 1 & 5 \\ 3 & 6 & -1 & -1 & 4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 3 & 3 & -2 \\ 0 & 1 & 2 & 2 & -5 \\ 0 & -3 & 4 & 4 & -4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & 4 & -2 \\ 0 & -1 & 3 & 3 & -2 \\ 0 & 0 & 5 & 5 & -7 \\ 0 & 0 & -5 & -5 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{18}{5} \\ 0 & 1 & 0 & 0 & -\frac{11}{5} \\ 0 & 0 & 1 & 1 & -\frac{7}{5} \\ 0 & 0 & 0 & 0 & 5 \end{array}\right)$$

This augmented matrix gives us the following equations:

$$\begin{aligned} x_1 &= \frac{18}{5} \\ x_2 &= -\frac{11}{5} \\ x_3 + x_4 &= -\frac{7}{5} \\ 0 &= 5 \end{aligned}$$

Now since $0 \neq 5$, we see that the system is inconsistent.

Problem 3. Determine whether the following system is inconsistent. If not inconsistent and no free variables, find the unique solution. If there are free variables find all solutions (describe the set).

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

To find solution(s), we make an augmented matrix and reduce the matrix to rref

$$\left(\begin{array}{ccc|c} -1 & 2 & -1 & 2 \\ -2 & 2 & 1 & 4 \\ 3 & 2 & 2 & 5 \\ -3 & 8 & 5 & 17 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 2 & -1 & 2 \\ -2 & 2 & 1 & 4 \\ 3 & 2 & 2 & 5 \\ 0 & 10 & 7 & 22 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 2 & -1 & 2 \\ 0 & -2 & 3 & 0 \\ 0 & 8 & -1 & 11 \\ 0 & 10 & 7 & 22 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 2 & -1 & 2 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 11 & 11 \\ 0 & 10 & 7 & 22 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 2 & -1 & 2 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 11 & 11 \\ 0 & 0 & 22 & 22 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 2 & -1 & 2 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 0 & 2 & 2 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 0 & 2 & 2 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix}$$

Problem 4. Find the inverse matrix for the following matrices:

(a) Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

We can find the inverse of our matrix by augmenting our matrix with the identity matrix:

$$\left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

(b) Let $B = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

$$\begin{aligned}
&\left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \\
=&\left(\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 3 & 0 & 1 \end{array}\right) \\
=&\left(\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 0 & 3 & -5 \end{array}\right) \\
=&\left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & -1 & 1 & -2 \end{array}\right) \\
=&\left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right)
\end{aligned}$$

Thus, $B^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

(c) Let $C = \begin{pmatrix} 2 & 6 \\ 3 & 8 \end{pmatrix}$ For the 2×2 matrix, we see that:*

$$C^{-1} = \frac{1}{\det(C)} \begin{pmatrix} 8 & -6 \\ -3 & 2 \end{pmatrix}$$

$$C^{-1} = -\frac{1}{2} \begin{pmatrix} 8 & -6 \\ -3 & 2 \end{pmatrix} \qquad \text{Since } \det(C) = (8)(2) - (-6)(-3) = -2$$

$$C^{-1} = \begin{pmatrix} -4 & 3 \\ \frac{3}{2} & -1 \end{pmatrix}$$

(d) Let $D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

We can find the inverse of our matrix by augmenting our matrix with the identity matrix:

$$\begin{aligned}
&\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\
=&\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\
=&\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)
\end{aligned}$$

Thus $D^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(e) Let $E = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 5 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

*For a given 2×2 matrix C, if the inverse exists, the formula for it is: $C^{-1} = \frac{1}{\det(C)} \begin{pmatrix} b & -d \\ -c & a \end{pmatrix}$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 2 & 0 & 5 & 1 & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -5 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -5 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right)$$

Thus $E^{-1} = \begin{pmatrix} 3 & 0 & -5 \\ 0 & \frac{1}{3} & 0 \\ -1 & 0 & 2 \end{pmatrix}$

(f) Let $F = \begin{pmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & -5 & 2 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & -5 & 2 & 1 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 1 & 6 & -3 & 0 & -1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -15 & 8 & 0 & 3 \\ 0 & 1 & 6 & -3 & 0 & -1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 1 & 6 & -3 & 0 & -1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 5 & 0 & -3 & 6 & -5 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{6}{5} & -1 \\ 0 & 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & 0 \end{array}\right)$$

Thus $F^{-1} = \begin{pmatrix} 2 & -3 & 3 \\ -\frac{3}{5} & \frac{6}{5} & -1 \\ -\frac{2}{5} & -\frac{1}{5} & 0 \end{pmatrix}$

(g) Let $G = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ -1 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 & -1 \\ 0 & 0 & 2 & 3 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -2 & -1 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & \frac{1}{2} \end{array}\right)$$

Thus, $G^{-1} = \begin{pmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ -2 & -1 & -1 \\ \frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix}$

Problem 5. Evaluate the following determinants

(a)

$$\det(A) = \begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix}$$

$$\begin{aligned} \det(A) &= 4(1)(-4) + 3(2)5 + 0 - 0 - 4(2)(-1) - 3(3)(-4) \\ &= -16 + 30 + 8 + 36 \\ &= 58 \end{aligned}$$

(b)

$$\det(B) = \begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$$

$$\det(B) = (-1)^{(1+1)}2 \begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} + (-1)^{(1+2)}0 \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} + (-1)^{(1+3)}0 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 0 \\ 1 & 1 & 3 \end{vmatrix}$$

$$+(-1)^{(1+4)}1 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix}$$

$$\det(B) = 2 \begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} + -1 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix}$$

$$\begin{aligned} &= 2[(1)2(3) + 0 + 0 - 0 - 0 - 0] + -1[0 + (1)2(1) + 0 - 0 - 0 - (1)1(-2)] \\ &= 2[6] - 1[4] \\ &= 8 \end{aligned}$$

Problem 6. Use Cramer's rule to solve the following system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned}$$

Putting this into matrix form, we get:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Thus: } \det(A) &= 1(2)3 + 2(1)1 + 1(2)2 - 1(2)1 - 1(1)2 - 2(2)3 \\ \Rightarrow \det(A) &= 12 - 16 \\ \Rightarrow \det(A) &= -4 \end{aligned}$$

Cramer's Rule says that $x_i = \frac{|B_i|}{|A|}$, where $|B_i|$ is the determinant computed when sticking $\mathbf{b}$ into the ith column of the matrix A.

Now we solve for $|B_i|$:

$$B_1 = \begin{pmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{pmatrix}$$

$$\det(B_1) = 5(2)3 + 2(1)9 + 1(6)2 - 1(2)9 - 5(1)2 - 2(6)3$$
$$\Rightarrow \det(B_1) = -4$$

$$B_2 = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{pmatrix}$$

$$\det(B_2) = 1(6)3 + 5(1)1 + 1(2)9 - 1(6)1 - 1(1)9 - 5(2)3$$
$$\Rightarrow \det(B_2) = -4$$

$$B_3 = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{pmatrix}$$

$$\det(B_3) = 1(2)9 + 2(6)1 + 5(2)2 - 5(2)1 - 2(2)9 - 1(6)2$$
$$\Rightarrow \det(B_3) = -8$$

Solving for x_i, we get:

$$x_1 = \frac{|B_1|}{|A|} = \frac{-4}{-4} = 1$$
$$x_2 = \frac{|B_2|}{|A|} = \frac{-4}{-4} = 1$$
$$x_3 = \frac{|B_3|}{|A|} = \frac{-8}{-4} = 2$$

Thus:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Problem 7. Let A and B be 6×6 matrices, with $\det(A) = -10$ and $\det(B) = 5$. Use the properties of determinants to compute

(a) $\det(3A) = 3^6(\det(A))^{\star} = 729(\det(A))$
$= 729(-10)$
$= -7290$

(b) $\det(A^T B^{-1}) = (\det(A^T))(\det(B^{-1})$
$= (\det(A))(\det(B^{-1}))$
$= (\det(A))(\frac{1}{\det(B)})$
$= (-10)(\frac{1}{5})$
$= -2$

Problem 8. Prove that if A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
Since A is invertible, we see that the inverse of A is A^{-1} and $AA^{-1} = I$ by definition of inverse, then:

$$\begin{aligned} \det(I) &= \det(AA^{-1}) \\ &= \det(A)\det(A^{-1}) \\ 1 &= \det(A)\det(A^{-1}) && \text{Since } \det(I) = 1 \\ \Rightarrow \det(A) &= \frac{1}{\det(A^{-1})} \end{aligned}$$

Problem 9. Let A be a 3×3 matrix with $\det(A) = 5$. Find each of the following if possible.

(a) $\det(A^T) = \det(A)$
$= 5$

(b) $\det(A + I) \neq \det(A) + \det(I)$ in general, thus we cannot say anything about it.

(c) $\det(2A) = 2^3 \det(A)$
$= 8\det(A)$
$= 8(5)$
$= 40$

*We see that $\alpha A = (\alpha I_n)(A)$ where $\alpha \in \mathbb{R}$. We then see that $\det((\alpha I_n)(A)) = \det(\alpha I_n)\det(A) = \alpha^n \det(A)$ as the determinant of a diagonal matrix is the product of its main diagonal.

Problem 10. Let A be an invertible matrix. Prove that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$$

Hint: Consider a decomposition of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & ? \\ CA^{-1} & ? \end{pmatrix}\begin{pmatrix} ? & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

We see that our decomposed matrices coming from

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

are equal to:

$$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^{\star}$$

Now, finding the determinants of each matrix yields:

$$\begin{vmatrix} I & 0 \\ CA^{-1} & I \end{vmatrix} = \det(I)\det(I) = 1$$

$$\begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} = \det(A)\det(D - CA^{-1}B)$$

Now, since

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} I & 0 \\ CA^{-1} & I \end{vmatrix}\begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix}$$

Then

$$\begin{aligned} \begin{vmatrix} A & B \\ C & D \end{vmatrix} &= (1)\det(A)\det(D - CA^{-1}B) \\ &= \det(A)\det(D - CA^{-1}B) \end{aligned}$$

*Multiplying out the matrices, we see that:

$$\begin{pmatrix} A + 0 & B + 0 \\ CA^{-1}A + 0 & CA^{-1}B + D - CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$